

# Transition to the Fulde-Ferrel-Larkin-Ovchinnikov planar phase : a quasiclassical investigation with Fourier expansion

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We explore, in three spatial dimensions, the transition from the normal state to the Fulde-Ferrel-Larkin-Ovchinnikov superfluid phases. We restrict ourselves to the case of the 'planar' phase, where the order parameter depends only on a single spatial coordinate. We first show that, in the case of the simple Fulde-Ferrell phase, singularities occur at zero temperature in the free energy which prevents, at low temperature, a reliable use of an expansion in powers of the order parameter. We then introduce in the quasiclassical equations a Fourier expansion for the order parameter and the Green's functions, and we show that it converges quite rapidly to the exact solution. We finally implement numerically this method and find results in excellent agreement with the earlier work of Matsuo *et al.* In particular when the temperature is lowered from the tricritical point, the transition switches from first to second order. In the case of the first order transition, the spatial dependence of the order parameter at the transition is found to be always very nearly a pure cosine, although the maximum of its modulus may be comparable to the one of the uniform BCS phase.

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## I. INTRODUCTION

Despite being actively investigated for forty years the problem of the structure of the superconducting order parameter in very high fields is still the subject of intensive research. In the compounds of interest the coupling of the magnetic field to electronic spins can no longer be ignored, and the situation where it is the only relevant one has to be considered. In this case one faces the problem of pairing electrons, for which the spin up and spin down chemical potentials are not the same. This question has been addressed independently by Fulde and Ferrell [1] (FF) and by Larkin and Ovchinnikov [2] (LO), who proposed that the best order parameter corresponds to pairs formed with a non zero total momentum, in contrast to the standard situation of the BCS theory. It is worth noting that this kind of problem has been found recently to be quite relevant for ultracold atomic Fermi gases [3] as well as for the physics of neutron stars [4–6]. More specifically Larkin and Ovchinnikov [2] considered for the order parameter superpositions of different plane waves, corresponding physically to different pair total momentum. They investigated which superposition was favored near the transition at  $T = 0$ . Nevertheless they considered only a second order phase transition, which is not the most general situation as we will discuss below. Hence the question of the exact structure of the order parameter in the FFLO phases is still an open problem. Since most experiments identifying tentatively FFLO phases rest heavily on the theoretical analysis, this is also a problem with major experimental implications.

In a preceding paper [7] we have investigated analytically the transition to the FFLO phases in the vicinity of the tricritical point (TCP), where the FFLO transition line starts. This point is located at  $T_{\text{tcp}}/T_{c0} = 0.561$  where  $T_{c0}$  is the critical temperature for  $\bar{\mu} = 0$ , with  $2\bar{\mu} = \mu_{\uparrow} - \mu_{\downarrow}$  being the chemical potential difference between the two fermionic populations forming pairs, as for example spin up and down electrons (the corresponding effective field is  $\bar{\mu}_{\text{tcp}}/T_{c0} = 1.073$ ). In agreement with preceding numerical work [8–10] we have found that the transition is first order to an order parameter which is, to a very good precision, simply proportional to a one-dimensional 'planar' texture  $\Delta(\mathbf{r}) \sim \cos(\mathbf{q} \cdot \mathbf{r})$ . This order parameter is actually the one which has been shown by Larkin and Ovchinnikov [2] to be the most favorable for a (second order) transition at  $T = 0$ . Compared with other works we have been able to understand qualitatively and quantitatively the reasons which favor this order parameter with respect to all the other possible ones. Namely we have shown that a real order parameter is favored, and that, among these states, those with the smallest number of plane waves are preferred. This then leads to an order parameter with a  $\cos(\mathbf{q}_0 \cdot \mathbf{r})$  dependence, in agreement with preceding work.

A remarkable feature of the results is that the location in the  $\bar{\mu}, T$  plane of this first order transition toward the 'planar' order parameter is very near the standard FFLO second order phase transition. This is true not only near the TCP [7–9] but also almost down to  $T = 0$  [10] (actually the transition to the 'planar' order parameter goes back to a second order phase transition at low temperature in agreement with LO). This proximity of a second order transition may lead to believe that the order parameter  $\Delta$  is reasonably small at the first order transition. This is trivially valid near the TCP where a Landau-Ginzburg type expansion up to sixth order in order parameter  $\Delta$  could be performed

[7,8], but it is a tempting hypothesis even at lower temperature. This possibility has been explored by Houzet et al. [9]. They found problems in applying this scheme because, for the stablest phase, namely the planar one, the coefficient of the sixth order in  $\Delta$  changes sign when the temperature is lowered not much below the TCP. This leads to an instability and so the expansion up to sixth order in powers of  $\Delta$  becomes inconsistent.

We will first analyze this problem and show that it is already present in the simple case of the Fulde-Ferrell phase where the order parameter is given by  $\Delta(\mathbf{r}) = \Delta \exp(i\mathbf{q}\cdot\mathbf{r})$ . This analysis gives a clear hint that an expansion in powers of  $\Delta$  is going to fail anyway at low temperature. This suggest that one should avoid to perform such an expansion. Among the various possibilities for improving the situation, one of them is to remark that, at the transition, near the TCP, the actual order parameter is quite near a simple superposition of plane waves [7] even if the transition is first order (for a second order transition the order parameter is exactly such a superposition, as investigated [2] for example by Larkin and Ovchinnikov at  $T = 0$ ). So the power expansion near the TCP amounts also to keep only the lowest order in a Fourier expansion of the order parameter. This leads to look for a Fourier expansion in the equations instead of a  $\Delta$  expansion.

Since we want to deal with the full nonlinear, space-dependent problem, the convenient starting point is not Gorkov's equations, but rather the quasiclassical equations of Eilenberger [11], Larkin and Ovchinnikov [12]. Not only are those equations in their simplest form the most compact and convenient formulation of our problem, but a major advantage is that they can be extended in full generality to much more complex situations [13] and allow to formulate transport problems, including many-body effects, with the same level of efficiency. However since we have to deal with a comparatively simpler problem, we will use for simplicity in this paper the original formulation and notations of Eilenberger [11]. In comparison the general formalism is used by Burkardt and Rainer [14] for an analysis of a FFLO transition in two dimensions with a planar order parameter.

In this paper we will show that the introduction of a Fourier expansion in the quasiclassical equations allows to obtain a solution which converges very rapidly toward the exact result. As a consequence a few terms in the expansion provide an excellent approximation. Here we will just deal with the principle of this method and its application to the planar transition. In particular we will rederive the results of Matsuo et al [10]. Applications to other more complex cases, which are the more fundamental interest of this procedure, will be considered in another paper.

The paper is organized as follows. In the next section we consider the free energy and study in particular the simple case of the Fulde-Ferrell phase, and show that it has singularities at  $T = 0$  which make an expansion in powers of the order parameter unreliable. In section III we explain our Fourier expansion for the simplest case of a cosine order parameter. This is then generalized in the following section to any (one-dimensional) order parameter. Finally we give in section V the results of the numerical implementation of our method.

## II. THE FULDE-FERRELL PHASE $T = 0$ FREE ENERGY

We will show that the problems arising in the expansion of the free energy in powers of the order parameter are already present when one considers the simple Fulde-Ferrell (FF) state. Let us start with the completely general expression, that we will use further on, for the free energy difference per unit volume between the superconducting state and the normal state [14–16] :

$$\Omega_s - \Omega_n = \int d\mathbf{r} \frac{1}{V} |\Delta(\mathbf{r})|^2 + 4\pi T N_0 \text{Re} \sum_{n=0}^{\infty} \int_{\bar{\omega}_n}^{\infty} d\omega \int \frac{d\Omega_k}{4\pi} [g_s(\omega, \hat{\mathbf{k}}, \mathbf{r}) - g_n(\omega, \hat{\mathbf{k}}, \mathbf{r})] \quad (1)$$

Here  $V$  is the standard BCS interaction and  $N_0$  is the single spin density of states at the Fermi energy. The difference  $\mu_{\uparrow} - \mu_{\downarrow} = 2\bar{\mu}$  between spin up and spin down chemical potentials comes in the definition of  $\bar{\omega}_n = \omega_n - i\bar{\mu}$  where  $\omega_n = \pi T(2n + 1)$  are Matsubara frequencies. For the "  $\xi$ -integrated " or quasiclassical Green's functions we have used Eilenberger's notations  $g(\omega, \hat{\mathbf{k}}, \mathbf{r}) = \frac{i}{\pi} \int d\xi_k G(\omega, \mathbf{k}, \mathbf{r})$  where  $\xi_k$  is the kinetic energy measured from the average Fermi level  $(1/2)(\mu_{\uparrow} + \mu_{\downarrow})$  and  $G(\omega_n, \mathbf{k}, \mathbf{r})$  is the usual temperature Green's function (these Green's functions we deal with are those for up spin, the down spin Green's functions are obtained by a simple transform and the sum over the spin leads to take the real part in Eq.(1)). With these notations we have  $g_n(\omega_n, \hat{\mathbf{k}}, \mathbf{r}) = 1$  for  $\omega_n > 0$ . It results directly from the starting Gorkovs equations [2] that the Green's functions in the presence of the effective field  $\bar{\mu}$  are obtained from those in the absence of it by the simple replacement of  $\omega_n$  by  $\bar{\omega}_n$ .

We look now for the expression of this free energy at  $T = 0$  for the simple FF state where the order parameter is given by  $\Delta(\mathbf{r}) = \Delta \exp(i\mathbf{q}\cdot\mathbf{r})$ . Since in this case  $|\Delta(\mathbf{r})|^2 = \Delta^2$ , the Green's function is just obtained from the standard BCS one by shifting [2] all the momenta by  $\mathbf{q}/2$ . Finally the quasiclassical Green's function is just the BCS one, except that we have to change  $\omega$  into  $\omega - i\bar{\mu}_k$  with  $\bar{\mu}_k = \bar{\mu}(1 - \bar{q}\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})$ , where we have defined the reduced wavevector  $\bar{q} = qk_F/(2m\bar{\mu})$  (this results also from the general Eilenberger's equations we will write below).

The free energy for the standard uniform BCS phase with  $\bar{\mu} = 0$  is :

$$\Omega \equiv \Omega_s - \Omega_n = \frac{1}{2} N_0 \Delta_0^2 x^2 \ln(x^2/e) \quad (2)$$

where  $\Delta_0 = 2\omega_c \exp(-1/N_0 V)$  is the zero temperature BCS phase gap ( $\omega_c$  is the standard cut-off of BCS theory), and we have expressed  $\Delta$  in units of  $\Delta_0$  by introducing  $x = \Delta/\Delta_0$ . This free energy is naturally minimum for  $x = 1$  and the minimum is  $-\frac{1}{2} N_0 \Delta_0^2$ . In the presence of a nonzero effective field  $\bar{\mu} > 0$  this expression becomes from Eq.(1):

$$\frac{\Omega}{N_0 \Delta_0^2} = -\frac{x^2}{2} + \bar{m}^2 + \text{Re}[x^2 \ln(\bar{m} + \sqrt{\bar{m}^2 - x^2}) - \bar{m} \sqrt{\bar{m}^2 - x^2}] \quad (3)$$

where we have also expressed  $\bar{\mu}$  in units of  $\Delta_0$  by setting  $\bar{m} = \bar{\mu}/\Delta_0$ . For  $\Delta > \bar{\mu}$  this free energy reduces to  $\frac{\Omega}{N_0 \Delta_0^2} = \frac{1}{2} x^2 \ln(x^2/e) + \bar{m}^2$  and gives the standard Clogston-Chandrasekhar [17,18] first order transition  $\bar{m} = 1/\sqrt{2}$ . On the other hand it gives for small  $\Delta \ll \bar{\mu}$  the expansion  $\frac{\Omega}{N_0 \Delta_0^2} = x^2 \ln(2\bar{m}) - x^4/8\bar{m}^2 - x^6/32\bar{m}^4$ , leading in particular to the second order spinodal transition for  $\bar{m} = 1/2$ . This expansion can be generalized at  $T \neq 0$  as :

$$\frac{\Omega}{N_0} = \ln[T/T_{sp}(\bar{\mu}/T)] \Delta^2 + \sum_{p=1}^{\infty} (-1)^{p+1} \frac{(2p)!}{2^{2p} p! (p+1)!} A_{2p} \Delta^{2p+2} \quad (4)$$

with :

$$A_{2p} = 2\pi T \text{Re} \left[ \sum_{n=0}^{\infty} \frac{1}{\bar{\omega}_n^{2p+1}} \right] \quad (5)$$

and  $T_{sp}(\bar{\mu}/T)$  is the temperature of the second order spinodal transition toward the standard BCS phase. It is interesting to note that, while the coefficients  $A_{2p}$  are clearly all positive when  $\bar{\mu} \rightarrow 0$ , they are given by  $A_{2p} = (-1)^p/(2p\bar{\mu}^{2p})$  when  $T \rightarrow 0$ . Moreover one can see that  $A_{2p}$  has  $p$  zeros when  $\bar{\mu}/T$  goes from 0 to  $\infty$  (one goes basically from  $A_{2p}$  to  $A_{2p+2}$  by taking a double derivative with respect to  $\bar{\mu}$ ). Hence the higher order coefficients have many changes of sign in the low temperature range. This feature corresponds to the singular behaviour which occurs for  $\Delta = \bar{\mu}$  at  $T = 0$  in Eq.(3). It allows also to understand that the change of signs found by Houzet et al. [9] are not simple accidents, but a systematic behaviour linked to the singularity appearing at  $T = 0$ .

Finally the  $T = 0$  free energy of the FF phase is obtained by replacing in Eq.(3)  $\bar{\mu}$  by  $\bar{\mu}_k = \bar{\mu}(1 - \bar{q} \hat{\mathbf{k}} \cdot \hat{\mathbf{q}})$  and averaging over the direction  $\hat{\mathbf{k}}$  as in Eq.(1). We give the result only in the case where  $\bar{q} > 1$  since this is the range of wavevector corresponding to the actual minimum of the free energy. One finds :

$$\begin{aligned} \frac{\Omega}{N_0 \Delta_0^2} = & -\frac{x^2}{2} + \bar{m}^2 \left(1 + \frac{\bar{q}^2}{3}\right) + \frac{x^2}{2\bar{m}\bar{q}} \text{Re}[\bar{m}_+ \ln(\bar{m}_+ + \sqrt{\bar{m}_+^2 - x^2}) - \sqrt{\bar{m}_+^2 - x^2} + (\bar{m}_+ \rightarrow \bar{m}_-)] \\ & - \frac{1}{6\bar{m}\bar{q}} \text{Re}[(\bar{m}_+^2 - x^2)^{3/2} + (\bar{m}_+ \rightarrow \bar{m}_-)] \end{aligned} \quad (6)$$

where we have used the notation  $\bar{m}_{\pm} \equiv \bar{m}(\bar{q} \pm 1)$ .

This result has singularities for  $\Delta = \bar{\mu}(\bar{q} \pm 1)$ . These are just the manifestation of the singularity found in Eq.(3) for  $\Delta = \bar{\mu}$ , corresponding to the upper and lower bound in the angular integration. In particular the singularity at  $\Delta = \bar{\mu}(\bar{q} - 1)$  gives the radius of convergence of the expansion in powers of  $\Delta$ . A particular consequence is that no expansion is possible for  $\bar{q} = 1$ . This is just the situation which is found when one works in a two-dimensional space. This singular situation leads to a cascade involving an infinite number of phase transitions when the temperature goes to zero, as we have shown elsewhere [19]. In the case of a three-dimensional space, with which we deal in this paper, the radius of convergence is non zero, but it is fairly small since the minimum free energy is found at low temperature for values of  $\bar{q}$  not far from the  $T = 0$  LO result  $\bar{q} = 1.2$ . Therefore a rapidly convergent expansion for the free energy is only valid for quite small  $\Delta$ , and this happens to be in contradiction with the values of  $\Delta$  needed to minimize this free energy. Naturally this expansion of Eq.(6) can be performed explicitly and the problem with the convergence is then quite obvious.

Now it is clear that these same problems arise if, instead of a phase with a single plane-wave as is the FF phase, we consider a more complicated phase which is a sum of plane waves, such as the planar phase  $\Delta(\mathbf{r}) \sim \cos(\mathbf{q} \cdot \mathbf{r})$ . This is already obvious from the fact that the terms which arise in the expansion for the FF phase will also appear in the expansion for this phase. Other terms with weaker singularities at  $\Delta = \bar{\mu}(\bar{q} - 1)$  will also be present. We note that

a singularity is already present in the fourth order terms investigated [2] by LO, as it can be seen from the explicit expression of their integral  $J$ , but it occurs for a specific value of the angle between the wavevectors which happens to be irrelevant for their final conclusion. Therefore we come to the conclusion that, due to the singular behaviour which occurs at  $T = 0$ , we can not rely anymore on an expansion in powers of  $\Delta$  when the temperature is lowered. It is conceivable that such an expansion could still be proper by accident for a specific phase, but it is unsafe for a general exploration of the various phases in competition. A possible partial cure for this problem could be to sum up the most divergent contributions, which are precisely those occurring in the FF phase. We have tried such an approach, but although it provides some improvement it clearly does not lead to a satisfactory situation.

Therefore in an attempt to extend the simple approach around the TCP we will in the next section proceed to a Fourier expansion in the exact quasiclassical formulation of the problem. This will prove to be completely satisfactory.

### III. FOURIER EXPANSION

We start from Eilenberger's equations for the diagonal  $g(\omega, \hat{\mathbf{k}}, \mathbf{r})$  and off-diagonal  $f(\omega, \hat{\mathbf{k}}, \mathbf{r})$  quasiclassical propagators, which we simplify from the outset by taking  $\hbar = 1$  and  $m = 1/2$ . They read [11] :

$$\begin{aligned}(\omega + \mathbf{k} \cdot \nabla) f(\omega, \hat{\mathbf{k}}, \mathbf{r}) &= \Delta(\mathbf{r}) g(\omega, \hat{\mathbf{k}}, \mathbf{r}) \\(\omega - \mathbf{k} \cdot \nabla) f^+(\omega, \hat{\mathbf{k}}, \mathbf{r}) &= \Delta^*(\mathbf{r}) g(\omega, \hat{\mathbf{k}}, \mathbf{r}) \\2\mathbf{k} \cdot \nabla g(\omega, \hat{\mathbf{k}}, \mathbf{r}) &= \Delta^*(\mathbf{r}) f(\omega, \hat{\mathbf{k}}, \mathbf{r}) - \Delta(\mathbf{r}) f^+(\omega, \hat{\mathbf{k}}, \mathbf{r})\end{aligned}\tag{7}$$

where  $\mathbf{k}$  is at the Fermi surface  $k = k_F$ . Actually  $g$  is given in terms of  $f$  and  $f^+$  by the normalization condition :

$$g(\omega, \hat{\mathbf{k}}, \mathbf{r}) = (1 - f(\omega, \hat{\mathbf{k}}, \mathbf{r}) f^+(\omega, \hat{\mathbf{k}}, \mathbf{r}))^{1/2}\tag{8}$$

so the last equation results from the two first ones. These ones are also related [11] since :

$$\begin{aligned}f^*(-\omega, \hat{\mathbf{k}}, \mathbf{r}) &= f^+(\omega, \hat{\mathbf{k}}, \mathbf{r}) & g^*(-\omega, \hat{\mathbf{k}}, \mathbf{r}) &= -g(\omega, \hat{\mathbf{k}}, \mathbf{r}) \\f^*(\omega, -\hat{\mathbf{k}}, \mathbf{r}) &= f^+(\omega, \hat{\mathbf{k}}, \mathbf{r}) & g^*(\omega, -\hat{\mathbf{k}}, \mathbf{r}) &= g(\omega, \hat{\mathbf{k}}, \mathbf{r})\end{aligned}\tag{9}$$

In this paper we consider only an order parameter which varies only along the  $x$  axis. Accordingly  $f$  and  $g$  depend only on this variable. Moreover we assume that the order parameter is periodic in this direction which is the situation occurring in the FFLO transition. We restrict also ourselves to real order parameters since these have been found to correspond to the highest critical temperature in the vicinity of the TCP, and the LO solutions are also real, so this property is expected to be widely satisfied. Anyway the generalization to an intrinsically complex order parameter should not make much difficulties.

Then we proceed to a Fourier expansion of this order parameter. Let us first assume, in order to present our method in the simplest case, that only the lowest harmonic is relevant. This amounts to take :

$$\Delta(x) = 2\Delta \cos(qx)\tag{10}$$

We will consider at the end of the paper the general situation, but we will actually find that, for our problem, the actual order parameter at the transition is very nearly a simple cosine. For fixed  $\mathbf{k}$  Eilenberger's equations are a set of first order differential equations for the variation of the Green's functions along  $\mathbf{k}$ . So we take a reduced variable along this direction by setting  $\mathbf{r} = \mathbf{k}X$ , which gives  $\mathbf{k} \cdot \nabla = d/dX$  and  $\Delta(x) = 2\Delta \cos(QX)$  where we have introduced  $Q = k_F q \cos \theta$  with  $\theta$  the angle between  $\mathbf{k}$  and the  $x$  axis. Then we make a Fourier expansion of the Green's functions:

$$f(X) = \sum_n f_n e^{inQX} \quad f^+(X) = \sum_n f_n^+ e^{inQX} \quad g(X) = \sum_n g_n e^{inQX}\tag{11}$$

Explicit substitution of Eq.(11) in Eilenberger's equations Eq.(7) gives:

$$\begin{aligned}f_n &= \frac{\Delta}{\omega + inQ} (g_{n-1} + g_{n+1}) \\f_n^+ &= \frac{\Delta}{\omega - inQ} (g_{n-1} + g_{n+1}) \\g_n &= \frac{\Delta}{2inQ} (f_{n-1} + f_{n+1} - f_{n-1}^+ - f_{n+1}^+)\end{aligned}\tag{12}$$

The solution of these equations have simple symmetry properties, which can be checked directly for example by generating explicitly the solution by a perturbation expansion. Actually they arise quite generally from the fact that we deal with an order parameter which is real and even (i.e. parity is not broken). This is more conveniently seen by taking the case where  $\omega$  is real. However one has to keep in mind that we will deal finally with a complex  $\omega$ . Nevertheless the symmetry properties are still valid generally in this case.

For real order parameter  $f(X)$ ,  $f^+(X)$  and  $g(X)$  are real, which is consistent with Eilenberger's equations. This implies  $f_{-n} = f_n^*$ ,  $f_{-n}^+ = f_n^{+*}$  and  $g_{-n} = g_n^*$ . Moreover for an even order parameter, Eq.(7) are unchanged when  $(\hat{\mathbf{k}}, \mathbf{r})$  is changed into  $(-\hat{\mathbf{k}}, -\mathbf{r})$  which shows that  $f$ ,  $f^+$  and  $g$  are also unchanged. Hence from Eq.(9)  $f(-X) = f^+(X)$  and  $g(-X) = g(X)$ , which leads finally to  $f_n^+ = f_{-n}$  and  $g_n = g_{-n}$ .

It is then convenient to make explicit the relation between  $f_n$  and  $f_n^+$  by introducing  $d_n = (f_n - f_n^+)/2i$ , which gives  $f_n = (i - \omega/nQ)d_n$ . We have then for the two quantities  $g_n$  and  $d_n$  (which are real for real  $\omega$ ) the recursion relations :

$$\begin{aligned} d_n &= -\frac{nQ\Delta}{\omega^2 + n^2Q^2}(g_{n-1} + g_{n+1}) \\ g_n &= \frac{\Delta}{nQ}(d_{n-1} + d_{n+1}) \end{aligned} \quad (13)$$

It is clear from these equations that  $g_n \neq 0$  only for even  $n$ , and  $d_n \neq 0$  only for odd  $n$ , as it can be seen for example by generating the solution perturbatively. Moreover they satisfy  $g_{-n} = g_n$  and  $d_{-n} = -d_n$ . These equations are linear and must be supplemented by the normalization condition Eq.(8). The  $n = 0$  component is enough and it provides us precisely with the spatial integral  $g_0 = \int d\mathbf{r} g_s(\omega, \hat{\mathbf{k}}, \mathbf{r})$  which we need in Eq.(1) to calculate the free energy :

$$g_0^2 = 1 - \sum_{n \neq 0} (g_n g_{-n} + f_n f_{-n}^+) = 1 - \sum_{n=1}^{\infty} (2g_n^2 + f_n^2 + f_{-n}^2) \quad (14)$$

Now the interesting point is the large  $n$  behaviour of  $g_n$  and  $d_n$ . If for example we eliminate  $d_n$  in Eq.(13) we obtain a linear recursion relation which links  $g_{n+2}$  to  $g_n$  and  $g_{n-2}$ . Since  $g_{-n} = g_n$  we have only to consider  $n \geq 0$ , but this becomes  $n \geq 2$  when one takes into account that in Eq.(13) the relation for  $g_0$  is identically satisfied because  $d_{-1} = -d_1$ . The general solution of such a recursion relation is a linear combination of two independent solutions. The large  $n$  behaviour is found from the recursion relation, which for  $\Delta, |\omega| \ll |nQ|$  simplifies into  $\Delta^2(g_{n+2} + g_{n-2}) + n^2Q^2g_n = 0$ . One sees that this equation has very rapidly growing solutions satisfying  $g_{n+2} \gg g_n \gg g_{n-2}$  and behaving as  $g_{2p+2} \sim (-1)^p(2Q/\Delta)^{2p}(p!)^2$ . Naturally these solutions are not physically acceptable. On the other hand the recursion relation has also a solution satisfying  $g_{n+2} \ll g_n \ll g_{n-2}$  and behaving as  $g_{2p} \sim (-1)^p(\Delta/2Q)^{2p}(1/p!)^2$ , which is the physical solution we are looking for. This solution is found only if  $g_0$  and  $g_2$  are related by a specific boundary condition.

The very fast decrease of  $g_n$  and  $d_n$  provides an easy way to obtain a set of approximate solutions, which moreover converges rapidly to the exact one, all the more since these are  $g_n^2$  and  $d_n^2$  which come in Eq.(14) for the calculation of  $g_0$ . Since  $g_n$  and  $d_n$  are very small for large  $n$  we just take them to be zero beyond some fixed value. This serves as boundary condition. Then we work backward to obtain the whole set of Fourier components and normalize them properly through the normalization condition Eq.(14). Specifically we proceed as follows. Since the recursion relations are linear we rescale  $g_n$  and  $d_n$  in order to have convenient initial values. We set  $g_n = g_0 G_n$  (which implies  $G_0 = 1$ ) and  $d_n = nQg_0 D_n$  and take as initial values  $G_{2N+2} = 0$  and  $D_{2N+1} \neq 0$  to be determined later. Then starting with  $p = N$  we use for decreasing values of  $p$  the following recursion obtained from Eq.(13) :

$$\begin{aligned} G_{2p} &= -G_{2p+2} - \frac{\omega^2 + (2p+1)^2Q^2}{\Delta} D_{2p+1} \\ D_{2p-1} &= \frac{1}{\Delta} \left[ \frac{2p}{2p-1} G_{2p} - \frac{2p+1}{2p-1} D_{2p+1} \right] \end{aligned} \quad (15)$$

down to  $G_0$ . All the  $G$ 's and  $D$ 's are proportional to  $D_{2N+1}$ , which is now found by enforcing  $G_0 = 1$ . Finally Eq.(14) gives explicitly for  $g_0$  :

$$g_0^{-2} = 1 + 2 \sum_{p=1}^N G_{2p}^2 + 2 \sum_{p=0}^N [\omega^2 - (2p+1)^2Q^2] D_{2p+1}^2 \quad (16)$$

When we let  $N \rightarrow \infty$  this equation provides the exact result for  $g_0$ . It is interesting to note that for these large  $n$  we have found that  $g_n$  is proportional to  $\Delta^n$ . This makes a precise link between the expansion in powers of  $\Delta$  we

discussed at the beginning and the Fourier expansion we are considering now. One can see our result as corresponding to resummations of infinite series, eliminating in this way the troubles mentioned in section II occurring because coefficients in the Landau-Ginzburg expansion change sign as the temperature is lowered. One finds also that in the limit of large  $|\omega| \gg \Delta, |nQ|$ , where one must recover the normal state Green's functions, one has  $g_{2p} \sim (-1)^p (\Delta/\omega)^{2p}$ . Naturally the recursion relations Eq.(15) are very convenient and very fast for a numerical implementation and in practice the situation is not very different from having an analytical expression for  $g_0$ . The only practical problem is linked to the determination of the square root in obtaining  $g_0$  from Eq.(16), but this is solved by noticing that, from the general spectral representation, one has  $\text{Re}g_0 \geq 0$  when  $\omega_n > 0$ .

The simplest of these approximations corresponds to take  $N = 0$  and it is given explicitly by :

$$g_0 = [1 + 2\Delta^2 \frac{\omega^2 - Q^2}{(\omega^2 + Q^2)^2}]^{-\frac{1}{2}} \quad (17)$$

This is already a quite non trivial approximation. Since it is correct up to order  $\Delta^2$  it gives the proper location for the standard FFLO second order transition line. Moreover as we will see it gives qualitatively and semiquantitatively the correct results, with a first order transition down from the TCP which becomes a second order transition at low temperature in agreement with Ref. [10].

Although it is quite simple the calculation of the free energy has to be carried out numerically and naturally it is the same for all the higher order approximations. In practice it is convenient to make use in Eq.(1) of  $\ln[T/T_{sp}(\bar{\mu}/T)] = 1/N_0 V - \pi T \sum \text{sgn}(\omega_n)/\bar{\omega}_n$  to rewrite it as :

$$\frac{\Omega_s - \Omega_n}{N_0} = \ln[T/T_{sp}(\bar{\mu}/T)] \int d\mathbf{r} |\Delta(\mathbf{r})|^2 + 4\pi T \sum_{n=0}^{\infty} \int_{\omega_n}^{\infty} d\omega \text{Re}[\langle g_0(\omega - i\bar{\mu}, \hat{\mathbf{k}}) \rangle_k - 1] + \frac{1}{2(\omega - i\bar{\mu})^2} \int d\mathbf{r} |\Delta(\mathbf{r})|^2 \quad (18)$$

where we have made no assumption on the spatial dependence of  $\Delta(\mathbf{r})$ . In the present case Eq.(10) gives  $\int d\mathbf{r} |\Delta(\mathbf{r})|^2 = 2\Delta^2$ . The form Eq.(18) is convenient for the frequency integration since the integrand behaves as  $\omega^{-4}$  for large  $\omega$ , with a corresponding behaviour  $\omega_n^{-3}$  in the Matsubara frequency summation. One may replace  $\ln[T/T_{sp}(\bar{\mu}/T)]$  by  $\ln[\bar{\mu}/\bar{\mu}_{sp}(\bar{\mu}/T)]$  where  $\bar{\mu}_{sp}(\bar{\mu}/T)$  is the spinodal field for a given  $\bar{\mu}/T$ , since  $\bar{\mu}_{sp}(\bar{\mu}/T)/T_{sp}(\bar{\mu}/T) = \bar{\mu}/T$ . Finally the angular average amounts to an integration over  $Q$  since  $\langle g_0(\hat{\mathbf{k}}) \rangle_k \equiv \int (d\Omega_k/4\pi) g_0(\hat{\mathbf{k}}) = \int_0^1 du g_0(Q = \bar{\mu}qu)$  with  $u = \cos \theta$  and  $\bar{q} = qk_F/\bar{\mu} \equiv \hbar qk_F/(2m\bar{\mu})$ .

Since we are only interested in the transition from the normal to the FFLO state, in order to obtain the corresponding critical temperature  $T_c$  we look for the highest temperature  $T$  (or effective field  $\bar{\mu}$ ) at which  $\Omega_s - \Omega_n = 0$ . Precisely this leads to the following equation for  $T = T_c$ :

$$\ln[T/T_{sp}(\bar{\mu}/T)] = -\text{Min} \frac{2\pi T}{\Delta^2} \sum_{n=0}^{\infty} \int_{\omega_n}^{\infty} d\omega \text{Re}[\langle g_0(\omega - i\bar{\mu}, \hat{\mathbf{k}}) \rangle_k - 1] + \frac{\Delta^2}{(\omega - i\bar{\mu})^2} \quad (19)$$

Since for homogeneity the right-hand side of this equation is only a function of  $T/\bar{\mu}$ ,  $\Delta/\bar{\mu}$  and  $\bar{q}/\bar{\mu}$  one has to minimize with respect to  $\Delta/\bar{\mu}$  and  $\bar{q}/\bar{\mu}$ , at fixed  $T/\bar{\mu}$ .

At  $T = 0$  this simplifies somewhat. The summation over Matsubara frequencies goes to an integral which is performed by a by parts integration. This gives for the critical field  $\bar{\mu}$  :

$$\ln[\bar{\mu}/\bar{\mu}_{sp}(T=0)] = -\text{Min} \frac{1}{\Delta^2} \int_0^{\infty} d\omega \omega \text{Re}[\langle g_0(\omega - i\bar{\mu}, \hat{\mathbf{k}}) \rangle_k - 1] + \frac{\Delta^2}{(\omega - i\bar{\mu})^2} \quad (20)$$

Finally we write down the self-consistency equation (or 'gap equation') for the order parameter, which comes out quite generally when the free energy is minimized with respect to variations of this order parameter. This equation gives the necessary feedback to Eq.(7) where  $\Delta(\mathbf{r})$  can not be naturally an open function. In practice we will make little use of it since we will rather minimize directly the free energy with respect to the order parameter. The self-consistency equation [11] can be written as:

$$\Delta_n \ln[T/T_{sp}(\bar{\mu}/T)] = 2\pi T \sum_{m=0}^{\infty} \text{Re}[\langle f_n(\omega - i\bar{\mu}, \hat{\mathbf{k}}) \rangle_k - \frac{\Delta_n}{\omega - i\bar{\mu}}] \quad (21)$$

by making the same transformation from  $T_{c0}$  to  $T_{sp}(\bar{\mu}/T)$  as we did to find Eq. (18).

#### IV. GENERAL ORDER PARAMETER

We consider now the extension of our Fourier expansion, presented in section III, to a general order parameter:

$$\Delta(x) = \sum_n \Delta_n e^{inQx} \quad (22)$$

We assume again a real order parameter, which implies  $\Delta_n^* = \Delta_{-n}$ . We also restrict ourselves as before to an order parameter even with respect to  $x$ , which makes  $\Delta_n$  real. Substitution as above in Eilenberger's equations give the following generalization of Eq.(12):

$$\begin{aligned} (\omega + inQ)f_n &= (\omega - inQ)f_n^+ = \sum_{p=1}^{\infty} \Delta_p (g_{n-p} + g_{n+p}) \\ 2inQg_n &= \sum_{p=1}^{\infty} \Delta_p (f_{n-p} + f_{n+p} - f_{n-p}^+ - f_{n+p}^+) \end{aligned} \quad (23)$$

Here we have also assumed for the moment that the spatial average of the order parameter is zero, that is  $\Delta_0 = 0$ . Introducing again  $d_n = (f_n - f_n^+)/2i$ , we obtain the recursion relations:

$$\begin{aligned} d_n &= -\frac{nQ}{\omega^2 + n^2Q^2} \sum_{p=1}^{\infty} \Delta_p (g_{n-p} + g_{n+p}) \\ g_n &= \frac{1}{nQ} \sum_{p=1}^{\infty} \Delta_p (d_{n-p} + d_{n+p}) \end{aligned} \quad (24)$$

We consider now the practical situation met in numerical use of these equations. In this case the number of Fourier components for the order parameter will be finite, so we have a maximum integer  $P$  for which  $\Delta_p = 0$  when  $p > P$ . We look again at the behaviour of the physical solution for  $d_n$  and  $g_n$  when  $n$  goes to infinity, and show that it is consistent with a fast factorial type decrease, as we have found in section III. Indeed in this case the dominant term in the sums found in the right hand side of Eq.(24) is the one where  $d_n$  or  $g_n$  has the smallest index  $n$ , which gives  $d_n \sim -\Delta_P g_{n-P}/(nQ)$  and  $g_n \sim \Delta_P d_{n-P}/(nQ)$ . This leads to  $g_n \sim (\Delta_P/Q)^{n/P}/(n!)^{1/P}$ . Although this is still a fast factorial type decrease, it gets slower when  $P$  increases. On the other hand the large  $n$  behaviour contains the power law dependence  $(\Delta_P)^{n/P}$ , where in generic situations  $\Delta_P$  is expected to be very small for large  $P$ . This is indeed what is found when one writes the self-consistency equation [11] Eq.(21) which gives  $\Delta_n$  in terms of  $f_n$ . This fast decrease of  $\Delta_n$  corresponds to the standard situation, where there is no smaller physical length scale for  $\Delta(x)$  than  $1/q$  itself. However one can think of other particular situations, where this fast decrease of  $\Delta_n$  does not hold and which should be dealt with specifically. Ultimately this convergence problem has to be handled numerically by making calculations for increasing  $P$  and looking when reasonable convergence has been achieved. This is what we will do below with our present problem of finding the location of the transition.

Finally we make the same practical use of this fast convergence property as in section III. We take as boundary condition that  $g_n$  and  $d_n$  are zero beyond some fixed value  $N$ . This allows to calculate all the  $g_n$  and  $d_n$  within a common multiplicative factor, which is then found by the normalization condition Eq.(14).  $N$  is progressively increased until convergence has been obtained. The situation for solving the practical problem of finding the  $g_n$ 's and the  $d_n$ 's is less convenient than in section III. However we still have a linear problem for which very efficient numerical procedures are known. We have basically to handle a matrix. Instead of having a tridiagonal matrix, with just matrix elements right below and above the main diagonal we have now a band diagonal matrix with, in addition to the main diagonal,  $2P$  diagonals with non zero matrix elements.

To be more specific we have now to take into account that, in our problem, only odd Fourier components  $\Delta_{2p+1}$  of the order parameter are nonzero. First we consider only order parameters with a zero spatial average  $\Delta_0 = 0$ , since taking a nonzero value amounts to mix in the order parameter of the uniform BCS phase, which is energetically unfavorable. Hence it is reasonable to assume that similarly a nonzero  $\Delta_0$  is unfavorable. Next we can see for example by a iterative treatment to all orders, in order to obtain an exact solution of Eilenberger's equations, that we have only odd components. Indeed if we start with the simple  $\Delta(x) = 2\Delta \cos(qx)$  that we considered in section III, we generate only odd Fourier components in  $f(X)$  and even components in  $g(X)$  as we have seen. Now this  $f(X)$  in turn generates only odd components for  $\Delta(x)$  from the self-consistency equation Eq.(21). But from Eq.(24) this is again completely compatible with only odd components for  $f(X)$  and even for  $g(X)$ . Naturally it can also be seen directly

from the starting Eilenberger's equations that such a solution is a consistent one. We note that such a solution with odd components means that, by shifting  $x$  by  $\pi/(2q)$ , we obtain an order parameter which is odd with respect to  $x$ , in the same way as it transforms Eq.(10) into  $2\Delta \sin(qx)$ .

Then it results from Eq.(24) that, just as in section III,  $g_n \neq 0$  only for even  $n$ , and  $d_n \neq 0$  only for odd  $n$ . In the same way we set  $g_n = g_0 G_n$  (implying  $G_0 = 1$ ) and  $d_n = nQg_0 D_n$ , and we have again  $G_{-n} = G_n$  and  $D_{-n} = D_n$ . It is now convenient to include  $g_n$  and  $d_n$  into a single unknown vector  $V_n$ , defined by  $V_{2p} = G_{2p}$  and  $V_{2p+1} = D_{2p+1}$ . Then Eq.(24) can be merely written as  $M_{mn}V_n = A_m$  with  $A_n = -\Delta_n$  and the matrix  $M$  given by:

$$\begin{aligned} M_{2n,2n} &= 2n \\ M_{2n+1,2n+1} &= \omega^2 + (2n+1)^2 Q^2 \\ M_{2m+1,2n} &= \Delta_{2(m+n)+1} + \Delta_{|2(m-n)+1|} \\ M_{2m,2n+1} &= (2n+1)[\Delta_{2(m+n)+1} - \Delta_{|2(n-m)+1|}] \end{aligned} \quad (25)$$

with  $m, n \geq 1$ .

As explained above we truncate the infinite matrix  $M$  by  $m, n \leq N_{\max}$ , which gives a matrix with finite order  $N_{\max}$ . The corresponding linear equation for  $V_n$ , with  $n \leq N_{\max}$ , can be solved numerically by efficient standard routines, since as mentioned above the matrix  $M$  has a generalized band diagonal form. Once this is done,  $g_0$  is still obtained from Eq.(16), the free energy calculated from Eq.(18) and the critical temperature obtained by minimization. Finally the whole procedure is repeated for increasing values of  $N_{\max}$  until convergence is achieved. In the next section we will display the corresponding numerical results.

## V. NUMERICAL RESULTS

We present now the results of our numerical implementation of the above procedure. In the first subsection below we restrict ourselves to an order parameter with only the lowest harmonic as it is given by Eq.(10). The general case is considered afterwards.

### A. Lowest harmonic

We first give in Fig. 1 the results for the calculation of the critical temperature, down from the TCP. Rather than giving  $T_c(\bar{\mu})$ , we cover for convenience the  $(\bar{\mu}, T)$  plane in polar coordinates, rather than cartesian coordinates, and give the critical temperature  $T_c(\bar{\mu}/T)$  as a function of  $\bar{\mu}/T$ , equivalent to a polar angle. More precisely we plot its ratio  $T_c/T_{FFLO}$  to the FFLO critical temperature  $T_{FFLO}(\bar{\mu}/T)$  obtained for the same value of the ratio  $\bar{\mu}/T$ . This is more suited to the present situation since we find this ratio to be always near unity. However to make the graph more readable, we give on the x-axis, instead of  $\bar{\mu}/T$ , the value of  $T_{FFLO}(\bar{\mu}/T)$  itself, compared to the standard BCS critical temperature  $T_{c0}$ , found for  $\bar{\mu} = 0$  (this corresponds to go along the standard FFLO transition line). Naturally when our result for  $T_c/T_{FFLO}$  goes below 1, this means that the first order transition is less favorable than the second order one, so when the temperature is lowered there is actually a switch from first to second order when one finds that  $T_c/T_{FFLO} = 1$ .

We give the results of the calculation with increasing values of  $N_{\max}$  going up to 6. The approximation  $N_{\max} = 1$  corresponds to the explicit result Eq.(17) for  $g_0$ . As already mentioned above, it is correct up to second order in  $\Delta$  and consequently it gives the correct location for the FFLO transition. Moreover we see that it gives already the proper result semi-quantitatively for the order of the transition, since it is gives a switch from first to second order when the temperature goes below  $T_{FFLO}/T_{c0} = 0.195$ . The next approximation  $N_{\max} = 3$  for odd  $N_{\max}$  is already quite good quantitatively since it gives 0.063 for the above ratio. Full convergence is obtained for  $N_{\max} = 5$  where we find  $T_{FFLO}/T_{c0} = 0.076$  in very good agreement with Matsuo *et al* [10]. For completeness we give also in Fig. 1 our results for even  $N_{\max}$ . It is less natural, from the structure of the recursion equations, to truncate them in this way. Hence it is not so surprising that the approximation  $N_{\max} = 2$  is much worse than  $N_{\max} = 1$  since it does not even give a switch from first to second order for the transition. Nevertheless we have naturally convergence when we increase  $N_{\max}$ , and indeed we find that  $N_{\max} = 4$  is already very good since the switch is located at  $T_{FFLO}/T_{c0} = 0.074$ , while  $N_{\max} = 6$  is completely converged.



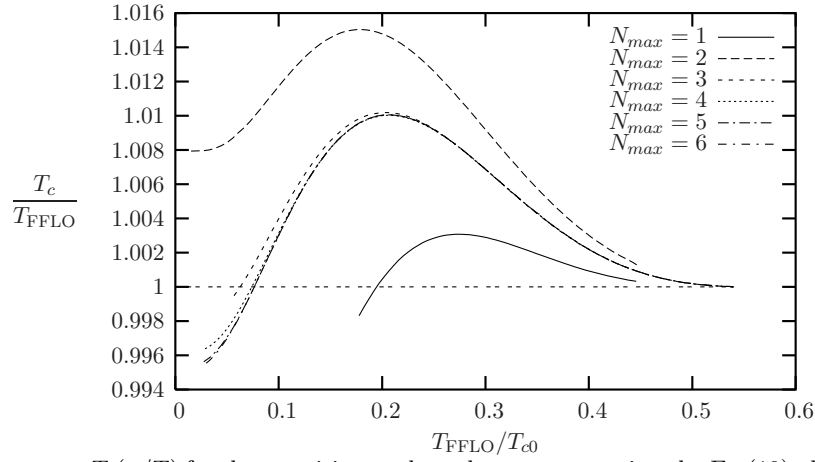


FIG. 1. Critical temperature  $T_c(\bar{\mu}/T)$  for the transition to the order parameter given by Eq.(10), divided by the FFLO critical temperature  $T_{FFLO}(\bar{\mu}/T)$  obtained for the same value of  $\bar{\mu}/T$ . On the  $x$  axis, instead of  $\bar{\mu}/T$ , we have given  $T_{FFLO}(\bar{\mu})/T_{c0}$ , where  $T_{c0}$  is the maximal critical temperature, obtained for  $\bar{\mu} = 0$ .

A noticeable feature of Fig. 1 is that the ratio  $T_c/T_{FFLO}$  stays always very near unity, while one would have expected to find it larger since there is no obvious relation between the order parameters of the first and the second order transitions. This behaviour is also found [7] near the TCP. A natural conclusion from this feature is to say that the first order transition is actually always quite near to be a second order one. We can check in our results if this interpretation is a coherent one by looking at the size of the order parameter (more precisely its maximal value as a function of spatial position), that is essentially the value of  $\Delta_1$  at the first order transition. If the first order transition is nearly a second order one, it should be small compared to a typical gap  $\Delta_0$  in the uniform BCS phase. Equivalently, in Fig. 2, we compare it to  $\bar{\mu}$  since it is of order  $\Delta_0$  in all the range we are interested in (at  $T = 0$  the FFLO result is  $\bar{\mu} = 0.754\Delta_0$ ). Our results show clearly that the size of the order parameter at the transition is of order of the one deep in the standard BCS phase, so it is not at all possible to consider that the first order transition is nearly a second order one.

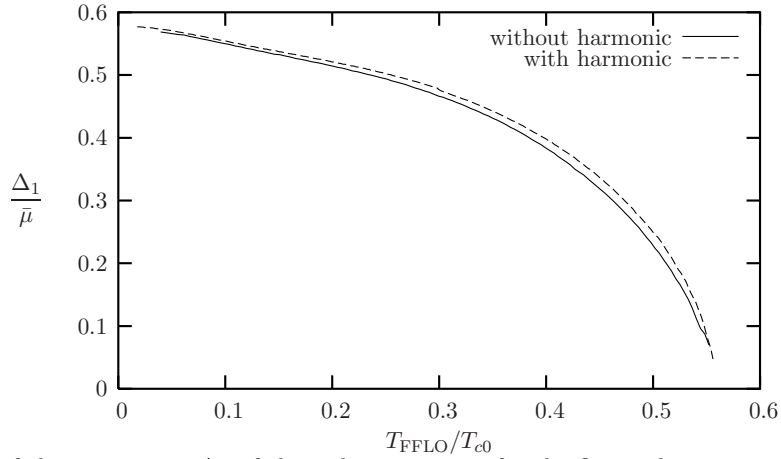


FIG. 2. Full line: size of the component  $\Delta_1$  of the order parameter for the first order transition, compared to  $\bar{\mu}$ , when all higher order harmonics are taken equal to zero. Dashed line: same quantity for an order parameter where  $\Delta_3$  is also non zero.

Finally it is also of interest to compare the reduced wavevectors of the order parameter for our first order transition and for the standard second order FFLO transition. This is done in Fig. 3 where it is seen that, although there are differences, they are not very large so that the reduced wavevector is rather similar for the two transitions, in contrast with the size of the order parameter.

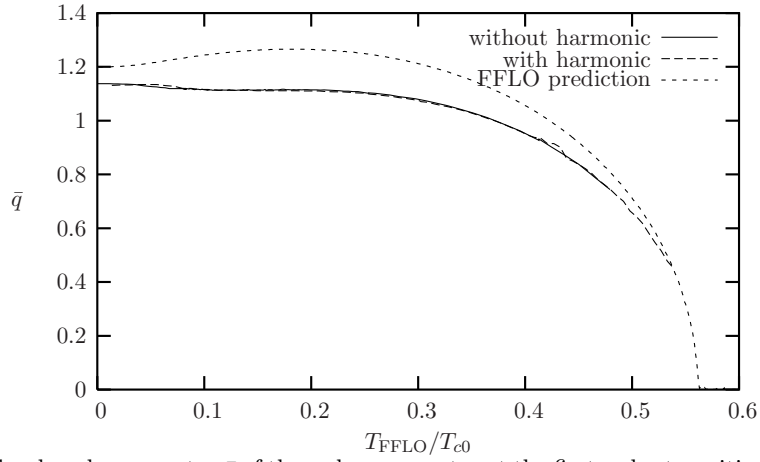


FIG. 3. Full line: optimal reduced wavevector  $\bar{q}$  of the order parameter at the first order transition, for the converged solution, as a function of temperature, when only the component  $\Delta_1$  is different from zero. Long dashed line: same result when  $\Delta_3$  is also non zero. Short dashed line: corresponding result for the second order FFLO transition

### B. Higher harmonics

Naturally it is not consistent to keep only the lowest harmonic in the order parameter, as it is immediately seen from the self-consistency equation (21). Hence we consider now the effect of higher harmonics. In a first step we have explored the effect at the transition of the inclusion of  $\Delta_3$ . We have found it quite small. This would imply normally to stop the exploration at this stage since one expects the effect of harmonics  $\Delta_5$  and higher to be even smaller. However one might wonder whether this result is not somewhat accidental [20] and specific to  $\Delta_3$ . This is especially a concern in the vicinity of the switching temperature from first to second order, where  $\Delta_3$  is particularly small (see Fig. 5). It could be that higher order harmonics dominate in this region leading to a quantitative change of the first order transition line. Hence, in order to eliminate any doubt on this question, we have explored the effect of taking  $\Delta_5$  and  $\Delta_7$  to be different from zero. Our study shows that these harmonics give also a very small contribution. Hence our conclusion is that the optimal order parameter remains always very close to a simple cosine in the whole temperature range from the tricritical point down to zero temperature.

Our numerical procedure is to use directly the free energy Eq.(18) by taking as an ansatz our form for the order parameter, with either three or four odd Fourier components. More precisely we maximize, with respect to  $\bar{q}, \Delta_i$  with  $i = 1, 3, 5, 7$ , the critical temperature from the generalization to our case of Eq.(19), as explained in section III. We then check that our optimal form satisfies the gap equation. We have also performed calculations by making use only of the gap equation. The results are not significantly different from the ones we display below, and most of the time agree with them within numerical accuracy. From a practical point of view, we have chosen high enough values of  $N_{\max}$ , typically  $N_{\max} = 12$ , so that numerical results are insensitive to changes in  $N_{\max}$ .

We give first in Fig. 4 our result for the critical temperature of the first order transition. The effect of all our higher order harmonics can be only barely seen in the figure, as compared to our calculation with only the lowest harmonics  $\Delta_1$ , already given in Fig. 1.

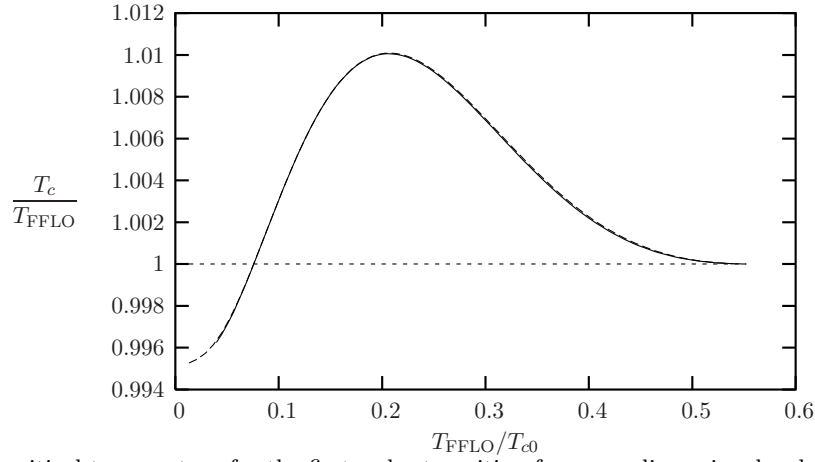


FIG. 4. Dashed line: critical temperature for the first order transition for a one dimensional order parameter form with four odd Fourier component  $\Delta_i$  with  $i = 1, 3, 5$  and  $7$ . Full line: same result for the simple cosine ansatz, where only  $\Delta_1$  is different from zero.

Next we display in Fig. 5, as a function of temperature, the values of the higher order harmonics  $\Delta_3, \Delta_5$  and  $\Delta_7$  for the optimal order parameter. It is seen that they are always quite small compared to  $\Delta_1$ . Nevertheless, around and below the switching temperature,  $\Delta_3$  and  $\Delta_5$  are of the same order while one would have rather expected  $\Delta_5$  to be small compared to  $\Delta_3$  (note that anyway these results are physically irrelevant below the switching temperature since they are for the first order transition, while the actual transition is second order). On the other hand  $\Delta_7$  is always negligible compared to  $\Delta_3$  and  $\Delta_5$ , except near the TCP where anyway  $\Delta_5$  and  $\Delta_7$  are essentially zero. Finally Fig. 5 shows also the results for the optimal  $\Delta_3$  and  $\Delta_5$  when  $\Delta_7 = 0$ . The difference with the preceding results is within numerical error. Similarly our result for  $\Delta_3$  when  $\Delta_5 = \Delta_7 = 0$  (not shown) are also essentially indistinguishable from the result displayed in Fig. 5.

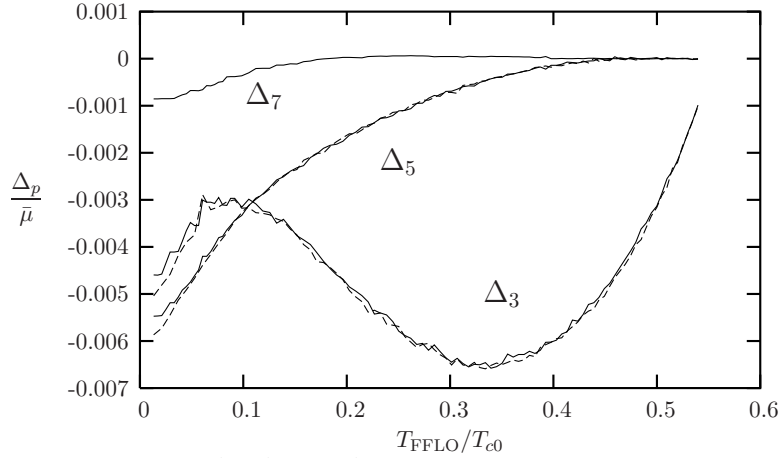


FIG. 5. Optimal values for the amplitudes  $\Delta_3, \Delta_5$  and  $\Delta_7$  in the order parameter as a function of temperature (full lines). For comparison the results for  $\Delta_3$  and  $\Delta_5$  when  $\Delta_7 = 0$  are also given as dashed lines.

## VI. CONCLUSION

In this paper we have shown that performing a Fourier expansion in the quasiclassical Eilenberger's equations provides a very efficient way to study the transition from the normal state to the FFLO phases in 3 dimensions, at least in the vicinity of the transition. We have applied this technique to the case of the transition to the one-dimensional 'planar' order parameter and we have found perfect agreement with the earlier work of Matsuo *et al.* In particular we have rederived their remarkable result that, when the temperature is lowered, the transition switches from first to second order. We have shown in detail that, in the case of the first order transition, the order parameter

is nevertheless dominated by its lowest order Fourier component, in somewhat surprising contrast to what one might guess for such a transition. This feature contributes naturally to make our Fourier expansion very rapidly converging.

However the strength of our method is not so much displayed in this case of a one-dimensional order parameter. Its major interest is rather that our approach can be fairly easily generalized to more complex order parameters, with full three-dimensional spatial dependence. As shown by Larkin and Ovchinnikov these order parameters come in competition and, in the case of a first order transition, it is not clear that they are not more advantageous than the standard second order FFLO phase transition. We will indeed show, in forthcoming work, that this is the case at low temperature in 3 dimensions. Finally another interest of our approach is to provide some insight, even if approximate, in the analytical structure of the theory, as we have seen by providing explicit approximate analytical solutions. We believe that this might be helpful in a theoretical situation where the intrinsic non linearity of the equations forces mostly to purely numerical work.

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